Stability of An Iterative Dynamical System

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***Abstract*— Stability is a classical yet active research topic for dynamical systems. Certain operators such as de-noising filters, smoothing filters and many algorithms may be applied iteratively. In many cases, they can be modelled as a complex dynamical system. Due to the errors and noises in acquisition of data, the stability of analysis results is vital to the validity of the analysis. However, little is known about the stability of analysis results in these situations. In this paper, we propose a method for analyzing the stability under iterations of operator. First we give the definition of stability under iterations of operator. We model the dynamics as an complex dynamical system. We introduce the concepts of Fatou and Julia set. We establish the connection of stability to Fatou and Julia set. We define different concepts of quasi-stability including asymptotical, bounded quasi-stability, which generalize the notion of stability. We provide the necessary and sufficient condition for quasi-stability under iteration of affine operator. We present a few results for the quasi-stability based on the concept of Fatou and Julia Set. Finally, we provide the numerical example to illustrate the theory.**

I. Introduction

The stability of dynamical system is a classical yet still active research area [1], [2], [3]. The Lyapunov stability theorem provides the sufficient condition for solution of differential equations [4]. Operators as de-noising filters, smoothing filters and certain algorithms may be applied iteratively in process of the data [5]. Naturally, the important question that whether the analysis result is robust and stable to the perturbations of the data rises in this situation. Due to the noises and errors in acquisition of the data, the stability issue under the iterations of operators can not be neglected. However, because of the different assumptions of the model, many classical result of stability can not be applied easily in these cases.

The dynamical system in many of these situations may be modelled as the iteration of a complex operator $\mathbf{\Phi} : \mathbb{C}^N \to \mathbb{C}^N$, where $N \in \mathbb{N}$. The stability in this case is the issue whether a small enough perturbation of initial point will end up in a small change for the outcome under $\mathbf{\Phi}^{\circ n}$, i.e. n-th iteration of $\mathbf{\Phi}$. The stability of this model is closely related to the complex dynamics [6], [7], [8]. We will see later, the concepts of Julia and Fatou set play an important role in complex dynamics.

Signals are often represented in a discrete way by their natures. In order to employ the signal processing techniques, one need to map the symbolic signal into numerical domain. It is conceivable that different such mappings could lead to contradictory conclusions. Interestingly, the stability in this case is related to the mapping consistency problem [9]. The stability can be seen as an equivalence to the mapping consistency problem under the iteration of operator.

In this paper, we propose an approach for analyzing many stability problems encountered in signal processing where we have iteration of certain operator. In Section II, we provide the preliminaries for the paper. We first define the stability and introduce the concepts of Fatou and Julia Set. We introduce a few important properties of Fatou and Julia set. We establish the connection between stability between Fatou and Julia set. We also show the necessary and sufficient condition for stability under iteration of affine operator. In Section III, we introduce concepts of quasi-stability as a generalization of stability and present several theoretical results on quasi-stability. In particular, In Section IV, we present experimental results which illustrate the theoretical results in genomic signal processing. Finally, we provide a brief summary and discussion of our results in Section V.

II. Dynamical Model and Stability

We model the operator as a complex operator (function). Let $\mathbf{\Phi} : \mathbb{C}^N \to \mathbb{C}^N$ be a holomorphic (analytic) operator. The input signal can be embedded as a point z in $\mathbb{C}^N$. In the case of the symbolic sequence, for example, a symbolic data sequence $\{a_i\}_{i=0}^{n-1}$, where $a_i \in \mathcal{A}$. The set $\mathcal{A}$ is the alphabet. In order to apply the signal processing techniques, we need to use a map f from $\mathcal{A}^n$ to $\mathbb{C}^N$. For example, if we have a mapping method $\tilde{f} : \mathcal{A} \mapsto \mathbb{C}^k$, then it naturally induces the map $f : \{a_i\}_{i=0}^{n-1} \mapsto z,\ z \in \mathbb{C}^{nk}$, where $([z]_{jk+1}, [z]_{jk+2}, ..., [z]_{jk+k})^T = g(a_j),\ j = 0, 1, ..., n-1$. Therefore, for a given symbolic sequence and a mapping method f, the corresponding numerical sequence is a point in $\mathbb{C}^N$. We denote this point as z_f. We will call the collection of iterations under function composition $\{\mathbf{\Phi}^{\circ n}\}_{n=1}$ as the *dynamical system*. In this paper we will assume $\mathbf{\Phi}$ is a polynomial, i.e. $(\mathbf{\Phi}(z))_i = P_i(z_1, z_2, ..., z_N), i = 1, ..., N$, where P_i is a polynomial. Note that by Taylor's theorem we know that any holomorphic map can be approximated by polynomials.

The stability issue is equivalent to the question that whether small changes of the given input sequence will cause a small changes for the outcome. It has many different definitions. In this paper, we give the following definition of stability:

Definition 1: An iuput z_0 is (Lyapunov)-stable for the dynamical system, if for any $\delta > 0$, there exists $\epsilon > 0$ such that for any point z in the ball of radius ϵ, centered at z_0 we

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have

$$\|\mathbf{\Phi}^{\circ n}(z) - \mathbf{\Phi}^{\circ n}(z_0)\| < \delta, \forall n \in \mathbb{N} \tag{1}$$

In study of complex dynamics, the concepts of Fatou and Julia set play a fundamental role. There are several non-equivalent definitions of Fatou and Julia set in multi-dimensional complex space [6], [7], [8]. We will use the definition below. Before that, we first introduce the notion of *normality*.

Definition 2 ([10]): A collection of holomorphic map $\mathcal{F}$ is called normal if every infinite sequence of maps from $\mathcal{F}$ either has a locally uniformly convergent subsequence or a subsequence diverges locally uniformly, i.e., for any point, there exists an open neighborhood such that the subsequence converges or diverges uniformly on that open neighborhood.

Recall that a sequence of functions is locally uniformly convergent means that for any point in the domain, there exists an open neighborhood of that point such that the sequence of functions converge uniformly in that neighborhood.

Definition 3 ([6]): The domain of normality F of $\mathcal{F} = \{\mathbf{\Phi}^{\circ n}\}$ is called Fatou set. Its complement

$$J = \mathbb{C}^N \backslash F \tag{2}$$

is called Julia set.

We define the basin of infinity as set of all points which have norms go to infinity under iteration.

The connected components of Julia (Fatou) set are called Julia (Fatou) components.

We will see later the Julia set represents the chaotic behaved points and points in Fatou set show rational behavior. We can show the following properties of Fatou and Julia set.

Proposition 1: A point z is in Fatou set if z is in the basin of infinity B.

Proof: It is straightforward to see if $z \in B$, then there exists a neighborhood U_z of z, such that

$$\lim_{n\to\infty} \|\mathbf{\Phi}^{\circ n}(z)\| \to \infty, \forall z \in U_z \tag{3}$$

So we have $z \in F$. ■

Proposition 2: Fatou (Julia) component is invariant. i.e. the operator maps one component onto another component.

Proof: It follows from the fact $\mathcal{F} = \{\mathbf{\Phi}^{\circ n}\}$ and $\mathcal{F} = \{\mathbf{\Phi}^{\circ(n+1)}\}$ have the same domain of normality and $\mathbf{\Phi}$ is continuous. ■

For point z is in the basin of infinity, although theoretically we can examine the stability, however, from computational point of view, the point diverges very fast under polynomial iterations. After a few rounds of iterations, the numerical results will overflow. In this case, the stability or even analysis result turns out to be meaningless. We show the following results about the connection of stability to the Fatou set.

Theorem 1: If z is not in basin of infinity, then z is stable if and only if z is in Fatou set.

Proof: (sketch of proof) If z is stable, then $\mathcal{F} = \{\mathbf{\Phi}^{\circ n}\}$ is eqi-continuous. Followed by Arzelà-Ascoli theorem [10], we have $z \in F$.

Conversely, it is enough to show that if z is not stable, we will have $z \in J$. If it is not stable, we have ϵ_0 and $n(1) < n(2) < \dots$ such that for $\{\mathbf{\Phi}^{\circ n(j)}\}_{j=1}$, $\exists\{z_j\} \to z_0$, we have

$$|\mathbf{\Phi}^{\circ n(j)}(z_j) - \mathbf{\Phi}^{\circ n(j)}(z_0)| > \epsilon_0 \tag{4}$$

if $z \in F$, we have a neighborhood U of z, such that $\{\{\mathbf{\Phi}^{\circ n(j)}\}\}$ has a uniformly convergent subsequence. Let h denote the limit. Taking limit for (4), we have

$$|h(z_j) - h(z_0)| > \epsilon_0 \tag{5}$$

for j large enough. But this contradicts to g is continuous. So $z \in J$. ■

From theorem 1, we see that Fatou set represents the good-behaved points. On the contrary, for the input z in the Julia set, no matter how small the perturbation the input has, it will not result in arbitrary small changes for the output. In this case, the analysis result may not be trust due to the instability.

III. Quasi-Stability Under Iterations

In this section, we generalize the notion of stability by introducing the concepts of quasi-stability. We first introduce different concepts of quasi-stability.

Definition 4: For the input z_1 and z_2, we say z_1 and z_2 are asymptotically quasi-stable to each other, if

$$\lim_{n\to\infty} \|\mathbf{\Phi}^{\circ n}(z_1) - \mathbf{\Phi}^{\circ n}(z_2)\| = 0 \tag{6}$$

z_1 and z_2 are called M-boundedly quasi-stable, if

$$\sup_n \|\mathbf{\Phi}^{\circ n}(z_1) - \mathbf{\Phi}^{\circ n}(z_2)\| < M. \tag{7}$$

All the points which are quasi-stable with z will be called quasi-stable class of z.

We have the following proposition, which shows quasi-stability generalizes the notion of stability by allowing any form of perturbation.

Proposition 3: The input z_0 is stable, then for any $\delta > 0$, there exists an open neighborhood U, such that any $z \in U$ is δ-boundedly quasi-stable with z_0.

Proof: It simply follows from the definition of stability. ■

If a point z is stable, then there always exists an open neighborhood U of z such that any point in U is boundedly quasi-stable to z. However, the converse is not true in general.

We first investigate the simplest case where $\mathbf{\Phi}$ is affine. We show the following results for quasi-stability.

Theorem 2: If $\mathbf{\Phi}(z) = Az + b$, any input z is asymptotically quasi-stable to each other, if and only if the spectral radius $\rho(A) < 1$.

Any input z is boundedly quasi-stable to each other, if and only if either $\rho(A) < 1$ or $\rho(A) = 1$ and all the eigenvalues have index ≤ 1.

Proof: (Sketch of proof) If for any x and y, $\|\mathbf{\Phi}^{\circ n}(x) - \mathbf{\Phi}^{\circ n}(y)\| \to 0$ as $n \to \infty$, then we have

$$\lim_{r\to\infty} A^r = 0 \tag{8}$$

Denote the $n \times n$ Jordan matrix with diagonal 0 as J_n. Consider the Jordan canonical form of A.

$$UAU^{-1} = \oplus_{i=1}^{l} \oplus_{j=1}^{k} (\lambda_i I_{m_{ij}} + J_{m_{ij}}) \tag{9}$$

where m_{ij} is the size of (i,j) jordan block.

Using the matrix function [11], we have that

$$A^r = U^{-1}(\oplus_{i=1}^{l} \oplus_{j=1}^{k} (\sum_{p=0}^{m_{ij}-1} \binom{r}{p} \lambda^{r-k} J_{m_{ij}}^{p}))U \tag{10}$$

So we must have the necessary and sufficient condition that $\rho(A) < 1$.

If we have any two maps are boundedly equivalent, then we have

$$\|A^r\| < M \tag{11}$$

for any r.

Follow from (10), we have either $\rho(A) < 1$ or if for some $|\lambda_i| = 1$, we must have there is only one dimensional Jordan block for that λ_i, i.e. index$(A) = 1$ if $\rho(A) = 1$. ■

Though in applications, Euclidean metric is the most widely used metric. However, it is often easier to work with the Kobayashi metric [12] than the Euclidean metric for investigating the quasi-stability.

For complex manifold M, one can construct a differential metric $F_M : T(M) \to \mathbb{R}$.

Definition 5: $F_M(\xi_x) := \inf(\frac{1}{r} : \exists f : D(r) \to M$ such that $f(0) = x$ *and* $df(\frac{\partial}{\partial z})_0 = \xi_x)$, where $D(r)$ is the disk centered at origin with radius r in complex plane. $\frac{\partial}{\partial z}$ is the basis of differential.

Once we have this differential metric, we can construct a pseudo metric called *Kobayashi pseudo metric*.

Definition 6: $d_K(x, y) := \inf_\gamma \{\int_a^b F_M(\dot{\gamma}(t))dt\}$ where $\gamma : [a, b] \to M$ is a piecewise C^∞ curve connecting x and y.

A complex manifold is called *hyperbolic* if the Kobayashi pseudo metric d_K is a metric. [12] is referred for the details of construction and properties of Kobayashi metric and hyperbolic manifold. One remarkable property for hyperbolic manifold under Kobayashi metric is the non-increasing principle [12].

Theorem 3 (Non-increasing Principle): If M is hyperbolic, then for any holomorphic function $\mathbf{\Phi}$ we have,

$$d_K(\mathbf{\Phi}(x), \mathbf{\Phi}(y)) \leq d_K(x, y), x, y \in M \tag{12}$$

Braverman and Yampolsky [13] showed the Julia set of certain types of polynomial can not be computed by any Oracle Turing Machine. If one of the point is in Julia set, it may not computable to figure out the quasi-stable class of the given input z. Also as we explained in previous chapter, from computational point of view, inputs in the basin of infinity are not necessary to be considered. We will classify the inputs falling in all these situations as the *computationally chaotic class*. Therefore, the only interesting case left would be if both points are in Fatou set.

It is often that if one analysis results is constant multiple of another, i.e. just a constant times another result, then the two result is usually thought to contain the same information. In this case we can define an equivalent class to eliminate this duplication. We will consider the projective space $\mathbb{P}^{N-1}$ of $\mathbb{C}^N - 0$, where we define an equivalent relation such that $x \sim y$ if $x = cy$ where $c \neq 0$. For the non-degenerate homogenous polynomial from $\mathbb{C}^N$ to $\mathbb{C}^N$, it natually rises to map from $\mathbb{P}^{N-1}$ to $\mathbb{P}^{N-1}$ through the natural projection map $\pi : \mathbb{C}^N - 0 \to \mathbb{P}^{N-1}$. So from now on we will assume the underline space is $\mathbb{P}^{N-1}$.

Ueda showed the following important result about the Fatou component for non-degenerate homogenous polynomial [14].

Theorem 4: If $\mathbf{\Phi}$ is non-degenerate homogenous polynomial, i.e. $\mathbf{\Phi}$ is homogenous and $\mathbf{\Phi}^{-1}(0) = 0$, then its Fatou component is hyperbolic.

We establish the following result about the equivalence of Kobayashi metric to the Euclidean metric under certain cases.

Theorem 5: If $\mathbf{\Phi}$ is non-degenerate homogenous polynomial, z_1 and z_2 are in Fatou set, if $d_K(z_1, z_2) < M$ then z_1 and z_2 are M-boundedly quasi-stable under d_K metric. In particular if z_1 and z_2 are in the same Fatou component U where $\mathbf{\Phi}(U) = U$ and U is hyperbolic, then any two points z_1 and z_2 in this Fatou component are boundedly quasi-stable under Euclidean metric.

Proof: (Sketch of Proof) The first claim follows directly from the definition. Using the property the Kobayashi metric is continuous [12], we can show that the open subsets with respect to the d_K topology are also open in the Euclidean topology. For the converse, for a point x, choose a relatively compact neighborhood U of x, consider $r = \min_{y \in \partial U} \{d_K(x, y)\}$, notice $r > 0$ so the r ball in d_K topology in contained in U. So we have the Kobayashi topology is equivalent to Euclidean topology. Together with the non-increasing principle we proved the second claim. ■

For the asymptotical quasi-stability, we show the following result.

Theorem 6: If $\mathbf{\Phi}$ is non-degenerate homogenous polynomial, U is a Fatou component, and $\mathbf{\Phi}(U) = U$, if $d_K(\mathbf{\Phi}(x), \mathbf{\Phi}(y)) < d_K(x, y)$ for any distinct $x, y \in U$, then there exists a unique fixed point in U and any z_1 and z_2 in U are asymptotically quasi-stable.

Proof: (Sketch of proof) Choose a point p_0, let $p_n = \mathbf{\Phi}^{\circ n}(p_0)$. If $\lim_{n\to\infty} d_K(p_0, p_n) = \infty$, then $\forall q_0 \in B(p_0, r)$, by triangle inequality, we have $d_K(q_n, p_0) \geq d_K(p_n, p_0) - r \to \infty$. Get a contradiction.

So we must have $\lim_{n\to\infty} d_K(p_0, p_n) < M$. Therefore $\exists n(1) < n(2) < \cdots$, such that $p_{n(j)} \to \hat{p}$. Let $g_j = \mathbf{\Phi}^{\circ(n(j+1)-n(j))}$ and $r_j = d_K(\hat{p}, p_{n(j)})$. We have

$$d_K(g_j(\hat{p}), p_{n(j+1)}) \leq r_j \tag{13}$$

$$d_K(g_j(\hat{p}), \hat{p}) \leq r_j + r_{j+1} \tag{14}$$

If M is hyperbolic and paracompact then the collection of holomorphic maps $Hol(M, M)$ is a normal family [12]. Also By Stone's theorem that every metric space is paracompact [15]. We have $Hol(U, U)$ is a normal family.

Let g be the accumulation point of $\{g_j\}$. So we have

$$g(\hat{p}) = \hat{p} \tag{15}$$

We have

$$f(\hat{p}) = f(g(\hat{(p)})) = g(f(\hat{p})) \tag{16}$$

but g has only one fix point, so $f(\hat{p}) = \hat{p}$.

It also follows easily from the strictly deceasing that the fix point is an attracting point. ■

IV. Experimental Results

We conduct the experiments on two DNA sequences AD169 and rhodopsin gene sequence. We consider the operator $\mathbf{\Phi}$ as a non-linear square smoothing filter defined as follow,

$$\mathbf{\Phi}(z_1, z_2, ..., z_N) = (\frac{{z_1}^2 + {z_2}^2}{2}, ..., \frac{{z_i}^2 + {z_{i+1}}^2}{2}, ..., \frac{{z_N}^2}{2}) \tag{17}$$

The symbolic sequence is mapping according to the following map,

$$\tilde{f}(a) = \begin{cases} 1 & \text{if } a = \texttt{A} \\ -1 & \text{if } a = \texttt{T} \\ i & \text{if } a = \texttt{G} \\ -i & \text{if } a = \texttt{C} \end{cases} \tag{18}$$

This is one of the widely used mappings. We denote the induced mapping point as z_f.

In Fig. 1, we show the slices of Julia and Fatou set of $\mathbf{\Phi}$ at $(z, 1, 1, ..., 1)$, $(z, i, i, ..., i)$ and $(z, 0.25+0.75i, ..., 0.25+0.75i)$. Julia set commonly possesses a fractal shape and could be connected or disconnected.

It can be shown the Fatou component U containing origin satisfies all the assumptions in theorem 6. Therefore any two points in U are quasi-stable. We consider the following perturbed signal $\tilde{f}'$,

$$\tilde{f}' = \tilde{f} + \Delta z \tag{19}$$

where $\Delta z \in \mathbb{C}$. In Fig. 2, we show the slice of the Fatou component U with z_f at origin and Δz is varying in the ball of radius 0.1, centered at 0. The white area is in the Fatou component.

In Fig. 3, we show the asymptotically quasi-stable case. The Euclidean distance for two arbitrarily chosen inputs which are in the previous Fatou component U changes with the number of iterations for human gene AD169 sequence. As we can see the distance converges to 0 with the increase of number of iterations.

In Fig. 4, we show the non-quasi-stable case. The Euclidean distance for two non-quasi-stable inputs changes with the number of iterations. One is in the previous Fatou component U and the other is not. As we can see the distance diverges with the increase of number of iterations.

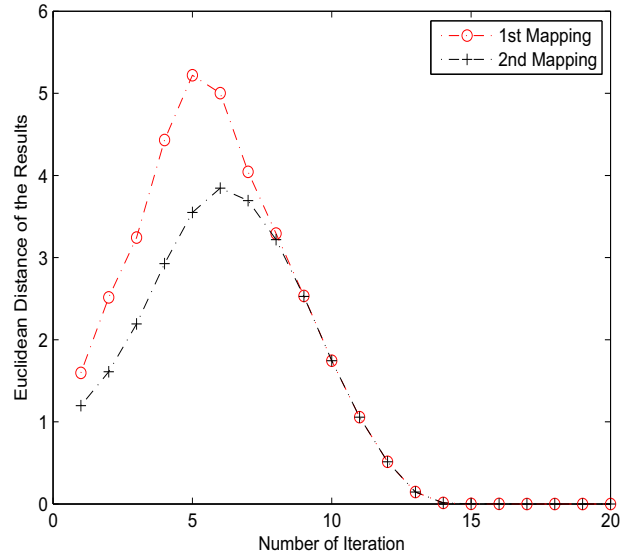


Fig. 3. Asymptotically quasi-stable case : The illustration of how Euclidean distance for two inputs which are in the previous Fatou component U changes with the number of iterations for human gene AD169 sequence.

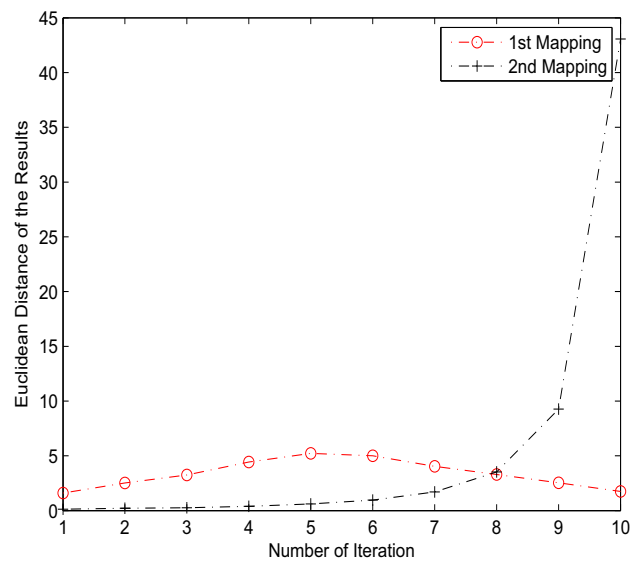
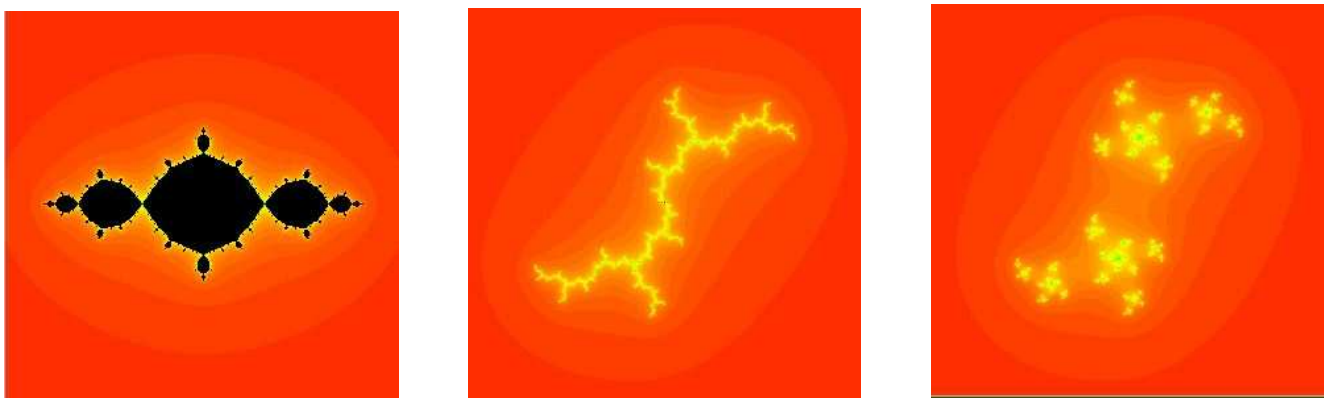
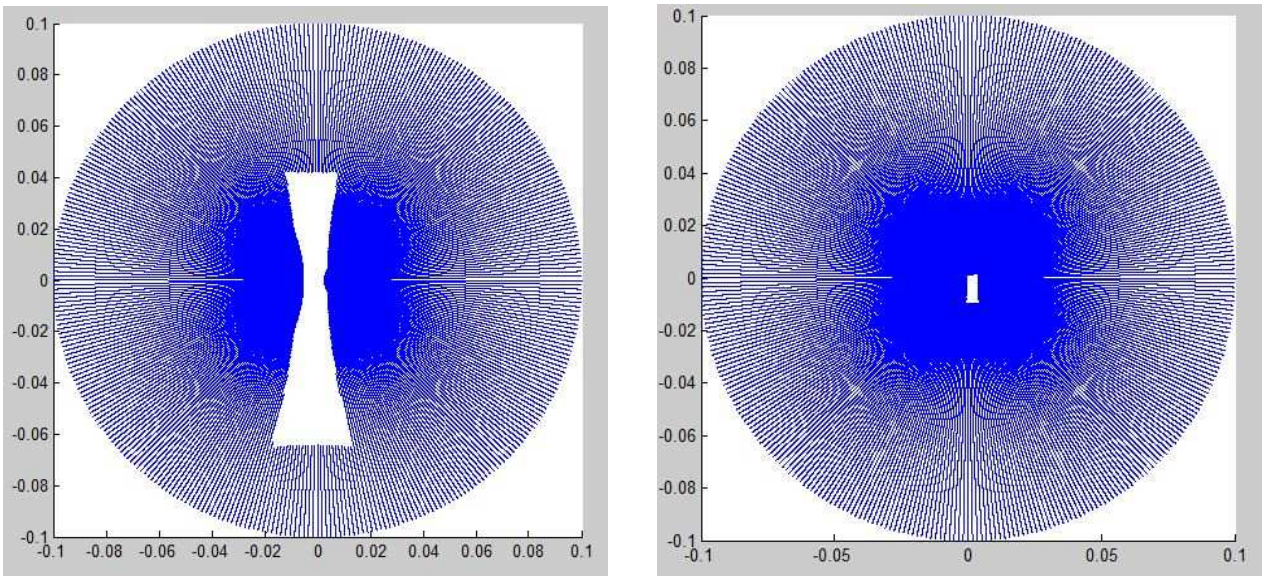


Fig. 4. Non-quasi-stable case: The illustration of how Euclidean distance for two inputs, for which one is in the previous Fatou component U and the other is not, changes with the number of iterations for human gene AD169 sequence.



(a) Julia Set and Fatou Set (b) Julia Set and Fatou Set (c) Julia Set and Fatou Set

Fig. 1. Slices of the Fatou and Julia Set of at $(z, 1, 1, ..., 1)$, $(z, i, i, ..., i)$ and $(z, 0.25+0.75i, ..., 0.25+0.75i)$ respectively. The Julia set is represented as the golden color. The red and black color represent the Fatou Set.



(a) Illustration of slice of the Fatou component U for human gene AD169 sequence.

(b) Illustration of slice of the Fatou component U for rhodopsin gene sequence.

Fig. 2. Quasi-stability analysis: (a), (b) show the slice of Fatou component U containing z_f for human gene AD169 sequence and rhodopsin gene sequence respectively. The origin represents z_f and the central white area is in U.

V. Conclusion

In this paper, we provide a method for analyzing stability of a dynamical system of iterations of operator. We give the definition of stability under iteration of operator. We also establish the connection of stability to Fatou and Julia set. We define different concepts of quasi-stability including asymptotical and bounded quasi-stability, which generalize the notion of stability. We provide the necessary and sufficient condition for quasi-stability under iterations of affine operator. We present a few results for the quasi-stability based on the concepts of Fatou and Julia Set. The Kobayashi metric is shown to be an important tool for investigating the stability. Finally, we conduct an experiment in genomic signal processing. We illustrate the stability and quasi-stability of a smoothing filter. In the future, we will study the dynamics where there is a composition of different operators.

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