

# FACTOR GRAPH-BASED STRUCTURAL EQUILIBRIA IN DYNAMICAL GAMES

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## ABSTRACT

Correlated equilibria are a generalization of Nash equilibria that permit agents to act in a correlated manner and can therefore, model learning in games. In this paper we define a special class of correlated equilibria that have hierarchical structure based on the factor graph. Such factor graph-based structural equilibria are more general than Nash equilibria and can model constrained dependencies than general correlated equilibria. We provide the numerical example for using non-cooperative stochastic game model on the gene regulatory network under three solution concepts.

**Index Terms**— Non-cooperative stochastic game, Factor graph-based structural equilibrium, Correlated equilibrium.

## 1. INTRODUCTION

Stochastic dynamical games have applications in cognitive radio systems, sensor networks, defense networks, gene regulatory network. [1, 2, 3, 4]. Although widely used, the Nash equilibrium has two disadvantages: The existence of Nash equilibrium requires use of fixed point theorems, and in general there are no provably convergent algorithms to a Nash equilibrium. Second, Nash equilibria assume that agents act independently, this is unrealistic in learning where agents interact with the environment and are therefore correlated. Hart and Mas-Colell observe in [5] that for most simple adaptive procedures, “... there is a natural coordination device: the common history, observed by all players. It is thus reasonable to expect that, at the end, independence among players will not obtain.” Harsanyi showed examples whose Nash equilibria are not reasonable even when they are unique [6].

Aumann proposed another solution concept called correlated equilibrium [5, 7], which generalizes the Nash equilibrium. The correlated equilibrium can be viewed as the result of Bayesian rationality, which can be interpreted as the distribution of play instruction given to players by some referee whose joint distribution is known to all. From a practical

view, the correlated equilibrium can be argued as the most relevant solution concept.

However, in many signal processing applications, the correlated equilibrium could be too “broad”. The feasible set comprises of many cases which do not fit the true rationale. The true correlations among the players could be neglected for the solutions.

In this paper we model the correlations among the players as a factor graph. Factor graphs are widely used in statistical signal processing. For example in LDPC code, Markov random field, etc [8, 9]. Since factor graphs are a useful representation of constrained dependencies amongst random variables, there is strong motivation to exploit factor graph structures in correlated equilibria in games. A factor graph is a bipartite graph representing the factorization of a function. In the case of dynamic game, the function is the joint distribution of the policies. As a graphical model, the dependence relations among random variables are denoted as a graph. The Markov network and Bayesian can be presented by the factor graph [9].

In this paper, we propose a new solution concept called factor graph-based structural equilibria which is a subclass of correlated equilibrium. The advantage of such a factor graph correlated equilibrium is that it allows modeling of constraints in the dependencies of agent behavior in the game. In section 2. we first provide the necessary background of the stochastic game and introduce the correlated equilibrium model. In section 3, We define the factor graph-based structural equilibria and illustrate some of its properties. In section 4, we provide the numerical example as gene regulatory network. Finally, we provide a brief summary and discussion of our results in Section 5.

## 2. NASH EQUILIBRIUM AND CORRELATED EQUILIBRIUM

We consider a Markovian dynamical game consists of state spaces  $S_t = \{s_i\}_{i=1}^n$  at each time index  $t$ . We assume there

are  $K$  players (agents). For each player  $k$ , it chooses action from its action set  $A_k(s)$ . We denote  $\mathbf{a} = \prod_{k=1}^K \mathbf{a}_k$  and  $\mathbf{A} = \prod_{k=1}^K A_k$ . At every epoch  $t$  the network updates and every player make decision at the same time.

Let  $\mathbf{s}_t$  denote the state of the system at time  $t$ . The system evolves as a controlled Markov process with transition probability,

$$\mathbb{P}\{s'|s, \mathbf{a}\} := \mathbb{P}\{\mathbf{s}_{t+1} = s' | \mathbf{s}_t = s, \mathbf{a}_1 = a^{(1)}, \dots, \mathbf{a}_k = a^{(k)}\} \quad (1)$$

In this paper, we always assume the dynamical game is of finite states and finite action sets. Notice that the transition probability of a Markovian dynamical game depends on the actions of all  $K$  players. The policy  $\pi_k = \{\pi_{k1}, \pi_{k2}, \dots\}$  for player  $k$  is a sequence of probability distributions such that  $\pi_{ki}$  is the decision distribution on the action set  $A_k$ . A policy is *pure*, if all the decisions in a policy are deterministic. If  $\pi_{ki}$  is independent of time  $i$ , then the policy is *stationary*. If the initial state  $\mathbf{s}_0$  is given, then it fixed a stochastic process. Let  $\mathbf{r}_t^k(\mathbf{s}_t, \mathbf{a}_1, \dots, \mathbf{a}_k)$  be the immediate reward function for player  $k$  at epoch  $t$ . So we have the expected reward function  $\mathbf{R}_t^k = \mathbb{E}_{\mathbf{s}_0}(\mathbf{r}_t^k)$ . In this paper, we mainly focus on the finite and infinite horizon discount average reward functions. For a give policy  $\pi$ , they are defined as,

$$V_k^L(\pi) := \sum_{n=1}^L \beta_k^{n-1} \mathbf{R}_n^k \quad (2)$$

$$V_k(\pi) := \sum_{n=1}^{\infty} \beta_k^{n-1} \mathbf{R}_n^k \quad (3)$$

Here  $\beta \in [0, 1)$  denotes the economic discount factor. The Correlated equilibrium and Nash equilibrium in the stochastic game are defined as follows,

**Definition 1.** A policy  $\pi^* = (\pi_1, \dots, \pi_K)$  is a *Correlated equilibrium point* if the following inequality holds,

$$V_k(\pi^*) \geq V_k(\mu_k, \pi_{-k}^*) \quad (4)$$

for any  $\mu_k \in \pi_k$  and any  $k \in K$ . The  $\pi_{-k}$  denotes the policies of all players except player  $k$ .

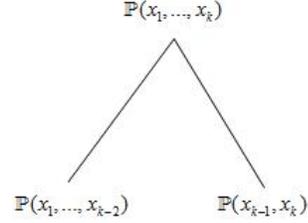
For a correlated equilibrium point  $\pi$ , if we have  $\pi = \prod_{k=1}^K \pi_k$ , then  $\pi$  is called a *Nash equilibrium*.

We denote collection of all the correlated equilibria as  $CE$ , the Nash equilibria as  $NE$ .

**Remark 1.** The set of Nash equilibria belongs to the set of correlated equilibria. Therefore existence of  $NE$  implies existence of  $CE$ . But in general the converse is not true.

### 3. THE FACTOR GRAPH-BASED STRUCTURAL EQUILIBRIUM

Although the correlated equilibrium allows for arbitrary dependencies of the elements of  $\pi$ , in many situations, we wish



**Fig. 1.** Factor graph of the case player 1 to  $k-2$  are correlated and independent of player  $k-1$  and  $k$

to specify the constraints on dependencies among different players. We need more refined description to model the constraints of these dependencies. For this reason we introduce the Factor Graph-based Structural Equilibria (FGSE).

The desired constraint on the dependencies is represented by a factor graph. For example, we may allow certain groups of players to be correlated. Among the correlated players, they may “cooperate” in some sense to achieve better outcome. For the joint distribution we could have the following decomposition using the Bayes’ rule.

$$\begin{aligned} \mathbb{P}(x_1, \dots, x_k) &= \mathbb{P}(x_1, \dots, x_{k-1} | x_k) \mathbb{P}(x_k) \\ &= \mathbb{P}(x_1, \dots, x_{k-2} | x_{k-1}, x_k) \mathbb{P}(x_{k-1} | x_k) \mathbb{P}(x_k) \\ &= \dots \end{aligned} \quad (5)$$

For example if we know player 1 to  $k-2$  are correlated and independent of player  $k-1$  and  $k$ , then we have the following decomposition.

$$\mathbb{P}(x_1, \dots, x_k) = \mathbb{P}(x_1, \dots, x_{k-2}) \mathbb{P}(x_{k-1}, x_k) \quad (6)$$

Or we could represent it in a factor graph as in Fig. 1.

We first give the definition of the factor graph-based structural equilibrium.

**Definition 2.** Given the factor graph of the joint distribution. Let  $\pi$  be a probability measure on  $\prod_{k=1}^K A_k$ .  $F$  be collection of all distribution having the given factor graph. The any policy in  $CE \cap F$  is called *factor graph-based structural equilibrium*. We denote collection of all the factor graph-based structural equilibrium as  $FGSE$

As we can see from the factor graph structure, we have the hierarchy of different levels of equilibrium. It is easy to have the following proposition for the factor graph-based structural equilibria.

**Proposition 1.**  $NE \subseteq FGSE \subseteq CE$ .  $NE = FGSE$  holds when  $\pi = \prod_{k=1}^K \pi_k$ .  $FGSE = CE$  holds when we have the trivial decomposition.

Gene	Predictors
CycD	Input
Rb	$(\overline{CycD} \wedge \overline{CycE} \wedge \overline{CycA} \wedge \overline{CycB})$
E2F	$(\overline{Rb} \wedge \overline{CycA} \wedge \overline{CycB})$
CycE	$(E2F \wedge \overline{Rb})$
CycA	$(E2F \wedge \overline{Rb} \wedge \overline{Cdc20} \wedge (\overline{Cdh1} \wedge \overline{Ubc})) \vee (CycA \wedge \overline{Rb} \wedge \overline{Cdc20} \wedge (\overline{Cdh1} \wedge \overline{Ubc}))$
Cdc20	$CycB$
Cdh1	$((\overline{CycA} \wedge \overline{CycB}) \vee \overline{Cdc20})$
Ubc	$(\overline{Cdh1}) \vee (Cdh1 \wedge Ubc \wedge Cdc20 \vee CycA \vee CycB)$
CycB	$(\overline{Cdc20} \wedge \overline{Cdh1})$

**Table 1.** Boolean functions of mammalian cell cycle.

*Proof.* The first claim follow from definition of FGSE and remark 1. When we have fully decomposition the NE corresponds to leaves of the factor graph. When we have trivial decomposition the only root corresponds to the CE.  $\square$

#### 4. CASE STUDY: GENE REGULATORY NETWORKS

The Stochastic game model with Nash equilibrium as solution concept has been applied to the gene regulatory network which is modelled as a probability Boolean network (PBN) [10]. The interactions of the regulations can be viewed as external controls on the transition probability [4].

In this section, we illustrate the factor graph-based structural equilibrium in an example comprising of the mammalian cell cycle with a mutated phenotype. The cycle regulation is proposed in [11]. We order these genes as  $s = \{CycD, Rb, E2F, CycE, CycA, Cdc20, Cdh1, Ubc, CycB\}$ . The state sets  $\{s\}$  can be interpreted as the binary expansion of  $\{0, 1, \dots, 511\}$ . Depending on the value of input CycD. We have two constituent Boolean networks of the PBN. We assume the two constituent networks have the same probability and the probability of switching is 0.01. In Table 1, we show the logic relation of all these genes.

We have the Rb, CycA and Ubc genes as the three controls  $\mathbf{a}_1$ ,  $\mathbf{a}_2$  and  $\mathbf{a}_3$  of the PBN respectively. We also assume the following reward functions.

$$\mathbf{r}^1 = \begin{cases} 10 & \text{if } \mathbf{a}_1 = 0 \text{ and } (CycD, Rb, Ubc) = (0, 0, 0) \\ 0 & \text{if } \mathbf{a}_2 = 1 \text{ and } (CycD, Rb, Ubc) \neq (0, 0, 0) \\ 2 & \text{otherwise} \end{cases} \quad (7)$$

$$\mathbf{r}^2 = \begin{cases} 9 & \text{if } \mathbf{a}_2 = 0 \text{ and } (CycD, CycA, Ubc) \neq (0, 0, 0) \\ 1 & \text{if } \mathbf{a}_3 = 1 \text{ and } (CycD, CycA, Ubc) = (0, 0, 0) \\ 3 & \text{otherwise} \end{cases} \quad (8)$$

$$\mathbf{r}^3 = \begin{cases} 5 & \text{if } \mathbf{a}_3 = 0 \text{ and } (CycD, CycA, Ubc) \neq (0, 0, 0) \\ 1 & \text{if } \mathbf{a}_3 = 1 \text{ and } (CycD, CycA, Ubc) = (0, 0, 0) \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

We consider the following decomposition of the joint distribution

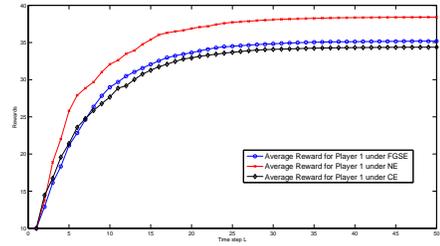
$$\mathbb{P}(x_1, x_2, x_3) = \mathbb{P}(x_1)\mathbb{P}(x_2, x_3) \quad (10)$$

In Fig. 2, Fig. 3 and Fig. 4, we show the average reward for three players under the Nash equilibrium, Correlated equilibrium and Factor graph-based structural equilibrium.

For player 1, we can see the NE behaves best among the three solution concepts, FGSE is better than CE. The intuitive reason behind this is that based on the assignments of the reward function, the reward for player 1 is contradictory to the other two players. The NE corresponds to the most ‘‘independent’’ case. In this case, player 2 and 3 do not work together in some sense. When in CE and FGSE, the player 2 and 3 are allowed to be correlated which result in the lower reward for player 1.

For player 2 and 3, we can see the CE and FGSE perform better than NE case. Since in the correlated case, the two players are allowed to ‘‘cooperate’’ to achieve better rewards than the NE case. Meanwhile the FGSE is better than CE. Because we constrain the specific dependencies structure by the factor graph, which will provide better result than CE case.

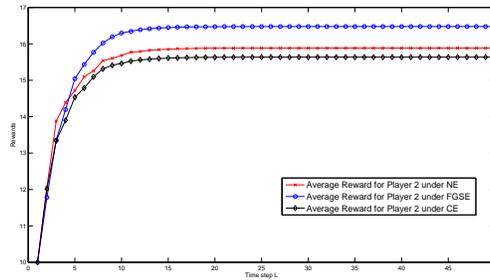
From the numerical experiment, we can see the FGSE captures more than the CE case if the correlations among the players are specified. The rewards for all the players are more relevant to the dependencies information provided by its factor graph.



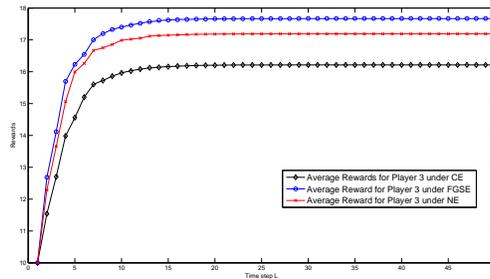
**Fig. 2.** The average reward for player 1 under NE, CE, FGSE

#### 5. CONCLUSION

In this paper, we propose a new solution concept called factor graph-based structural equilibria which captures more information than the Nash equilibrium. It also refines the solution set defined by the correlated equilibrium by combining the specified correlation information. The relationship of factor graph-based structural equilibria to the Nash equilibria and



**Fig. 3.** The average reward for player 2 under NE, CE, FGSE



**Fig. 4.** The average reward for player 3 under NE, CE, FGSE

Correlated equilibria has been shown. The FGSE as a solution concept fills the gap in NE and CE. The numerical example on gene regulatory network shows that FGSE is shown to provide more rational results for the given factor graph of the joint distribution.

## 6. REFERENCES

- [1] J.A. Filar, K. Vrieze, and OJ Vrieze, *Competitive Markov decision processes*, Springer Verlag, 1997.
- [2] J. W. Huang and V. Krishnamurthy, "Transmission control in cognitive radio as a markovian dynamic game: Structural result on randomized threshold policies," *IEEE Trans. on Communications*, vol. 58, no. 1, pp. 301–310, Jan. 2010.
- [3] M. Maskery and V. Krishnamurthy, "Cooperative engagement against anti-ship missiles: A correlated equilibrium game theoretic approach to netcentric force protection," *IEEE Trans. on Aerospace and Electronic Systems*, vol. 43, no. 7, pp. 843–863, July 2007.
- [4] L. Wang and D. Schonfeld, "Game theoretic model for control of gene regulatory networks," *IEEE International Conference on Acoustics Speech and Signal Processing (ICASSP)*, pp. 542–545, March 2010.
- [5] S. Hart and A. Mas-Colell, "A simple adaptive procedure leading to correlated equilibrium," *Econometrica*, vol. 68, no. 5, pp. 1127–1150, Sept. 2000.
- [6] J. C. Harsanyi, "Games with randomly disturbed payoffs: A new rationale for mixed strategy equilibrium points," *International Journal of Game Theory*, vol. 2, pp. 1–23, 1973.
- [7] R. J. Aumann, "Correlated equilibrium as an expression of bayesian rationality," *Econometrica*, vol. 55, no. 1, pp. 1–18, Jan. 1987.
- [8] R. Kindermann and J. L. Snell, *Markov Random Fields and Their Applications*, Amer. Math. Soc, Providence, RI, 1980.
- [9] F. Kschischang, B. Frey, and H. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. on Information Theory*, vol. 47, no. 2, pp. 498–519, Feb. 2001.
- [10] I. Shmulevich, E.R. Dougherty, S. Kim, and W. Zhang, "Probabilistic Boolean networks: a rule-based uncertainty model for gene regulatory networks," *Bioinformatics*, vol. 18, no. 2, pp. 261–274, 2002.
- [11] A. Faure, A. Naldi, C. Chaouiya, and D. Thieffry, "Dynamical analysis of a generic Boolean model for the control of the mammalian cell cycle," *Bioinformatics*, vol. 22, no. 14, 2006.