COMPRESSED SENSING: A NOVEL POLYNOMIAL COMPLEXITY SOLUTION TO NASH EQUILIBRIA IN DYNAMICAL GAMES

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ABSTRACT

Nash equilibria have been widely used in cognitive radio systems, sensor networks, defense networks and gene regulatory networks. Although solving Nash equilibria has been proved difficult in general, it is still desired to have algorithms for solving the Nash equilibria in various special cases. In this paper, we propose a compressed sensing based algorithm to solve Nash equilibria. Such compressed sensing method provides us a polynomial algorithm allowing more general payoff functions for certain classes of 2-player dynamic game. We also provide numerical examples to demonstrate the efficiency of proposed compressed sensing based method in solving Nash equilibria of 2-player games compared to existing algorithms.

***Index Terms*—** Nash equilibria, compressed sensing, dynamic game.

1. INTRODUCTION

Game theory is a mathematical model describes and analyzes scenarios with interactive decisions. In recent years, there has been a growing interest in adopting cooperative and noncooperative game theoretic approaches to model many communications and networking problems, such as cognitive radio systems, sensor networks, defense networks and gene regulatory networks [1, 2, 3]. Many of these applications employ solution concepts such as correlated equilibria and Nash equilibria. Nash equilibria can capture decision balance among all players at the expense of computation. Correlated equilibria extends the Nash equilibria and are benign to solve. Although widely used and computationally less expensive, the correlated equilibrium could be too "broad". Furthermore, the true correlations among the players could be neglected for the solutions. Compared with correlated equilibria, the Nash equilibria assume that agents act independently and have received great attention in the signal processing and communication communities [2]. The existence of Nash equilibrium requires essential use of Brouwer fixed point theorems [4] and in general, any algorithm solving the fixed point problem would unconditionally require an exponential number of function evaluations. The particular path following algorithm developed by Lemke and Howson [5] was recently proven to require, even in the best case for some instances, an exponential number of steps [6]. The general problem for solving Nash equilibria has been shown PPAD-complete [7], which is currently lack of general efficient algorithm. Instead, research attentions for efficient algorithm have been put on various special classes of the problem. For example, the problem of computing a Nash equilibrium in a two-player zero-sum game is solvable in polynomial time by Khachiyan's ellipsoid algorithm [8]. The problem with convex payoff functions can be solved by convex optimization tools such as interior point method [9]. However, it is still unknown whether we can have efficient algorithm if we have situations other than those special cases.

Compressed sensing is a signal processing technique for efficiently acquiring and reconstructing a signal, by finding solutions to under-determined linear systems. This takes advantage of the signal's sparseness in some domain, allowing the entire signal to be determined from relatively few measurements. Donoho showed that the number of linear equations can be small and still contain nearly all the information to reconstruct the signal [10].

In this paper, a compressed sensing based method is proposed to solve Nash equilibria. The method can be shown efficient for solving Nash equilibria for certain class of problem. In section 2, we first provide the necessary background of the compressed sensing theory. The essentials of the compressed sensing theory is that it allows the recovery of sparse solution to a under-determined linear equations system. In section 3, we make connections of solving Nash equilibrium to compressed sensing theory by using the fact that 2-player Nash equilibria reside in the vertexes of polytope formed by correlated equilibria. In section 4, we provide numerical examples to demonstrate the efficiency of proposed compressed sensing framework in solving Nash equilibria in 2-player games. Finally, we provide a brief summary and discussion of our results in section 5.

2. COMPRESSED SENSING APPROACH FOR UNDER-DETERMINED LINEAR EQUATIONS SYSTEM

In this section, we provide a brief review of the idea behind the compressed sensing theory. We use the term "signal" to represent the solution data we are trying to acquire. Let $x \in \mathbb{R}^n$ represent a signal and $y \in \mathbb{R}^m$ a vector of linear measurements formed by taking inner products of x with a set of linearly independent vectors $a_i \in \mathbb{R}^n$, $i = 1, 2, ..., m$. In matrix format, the measurement vector is $y = Ax$, where $A \in \mathbb{R}^{m\times n}$ has rows a_i^T, $i = 1, 2, ..., m$. When the number of measurements m is equal to n, the process of recovering x from the measurement vector y simply entails solving a linear equations system. However, in many applications, one only has very fewer measurements compared to a much larger dimension of space the signal x resides in, i.e., $m \ll n$. In that case, the linear system $Ax = y$ is typically under-determined, permitting infinitely many solutions. In order to have a unique solution, one need to apply various regulatory conditions.

In compressed sensing, one adds the constraint of sparsity, allowing only solutions to have smallest number of nonzero coefficients. Specifically, we are trying to solve the follow optimization problem.

$$\min\{\|x\|_0 : Ax = y\}, \tag{1}$$

where the quantity $\|x\|_0$ denotes the number of non-zeros entries in x. (1) is a combinatorial optimization problem with a prohibitive complexity if solved by enumeration, and thus is not tractable. An alternative model is to replace (1) by (2) and solve a computationally tractable convex optimization problem:

$$\min\{\|x\|_1 : Ax = y\}. \tag{2}$$

Under favorable conditions the combinational problem (1) and convex programming (2) share a common solution [11]. This equivalence result allows one to solve the L_1 problem, which is much easier than the original L_0 problem.

In this paper, we will employ the compressed sensing theory to solve the Nash equilibrium which can be formulated as solutions of a under-determined system of linear equations. More specifically, we use the basis pursuit model (2) for recovery represents a fundamental instance of compressed sensing. Certainly not the only one, many other recovery methods such as greedy-type algorithms are also available [12].

Theory of compressed sensing presently consists of two components: recoverability and stability. Recoverability addresses the central questions: what types of measurement matrices and recovery procedures ensure exact recovery of all k-sparse signals and how many measurements are sufficient to guarantee such a recovery? On the other hand, stability addresses the robustness issues in recovery when measurements are noisy and/or sparsity is inexact. We first review an important concept in compressed sensing.

Definition 1. *A measurement matrix A satisfies the* Restricted Isometry Property (RIP) *if the following inequality holds for all i-sparse vector x, $i \leq m$ and $\epsilon \in (0, 1)$*

$$(1 - \epsilon)\|x\|_2 \leq \|Ax\|_2 \leq (1 + \epsilon)\|x\|_2 \tag{3}$$

Recoverability is ensured if the matrix $A \in \mathbb{R}^{m\times n}$ holds RIP for certain pairs of (i, ϵ) [11].

Choosing a measurement matrix A with $m < n$ that has proper RIP ensures exact recovery of signal. In practice, it is almost always the case that either measurements or the measurement matrix is inexact, or both. The compressed sensing stability studies the issues concerning how accurately a compressed sensing approach can recover signals under these circumstances [13]. Stability results have been established for (2) and its extension

$$\min\{\|x\|_1 : \|Ax - y\|_2 \leq r\}. \tag{4}$$

Most compressed sensing methods have been shown to possess recoverability with known stability [12, 14].

3. COMPRESSED SENSING FRAMEWORK AND NASH EQUILIBRIUM

Let (S, u) be a game with n players, where S_i is the strategy set for player i, $S = S_1 \times S_2 ... \times S_n$ is the set of strategy profiles and u is the payoff function for $s \in S$. Let s_i be a strategy profile of player i and s_{-i} be a strategy profile of all players except for player i. When each player $i \in \{1, ..., n\}$ chooses their corresponding strategy s_i, which results in a strategy profile $s = (s_1, ..., s_n)$, then player i obtains his payoff $u_i(s)$. Note that the payoff of individual player depends on the strategy profile chosen by all players.

Definition 2. *A strategy vector $s \in S$ is said to be a* Nash equilibrium *if for all players i and each alternate strategy $s'_i \in S_i$, we have that*

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}). \tag{5}$$

In other words, no player i can change his chosen strategy from s_i to s'_i and thereby improve his payoff, assuming that all other players stick to the strategies they have chosen in s. The Nash equilibrium we have considered so far are called pure strategy equilibrium. For the notion of mixed Nash equilibrium, let us enhance the choices of players so each one can pick a probability distribution over his set of possible strategies; such a choice is called a *mixed strategy*.

A *correlated equilibrium* is a probability distribution over strategy vector s [15]. Let $p(s)$ denote the probability of strategy vector s, where we also use the notation $p(s) = p(s_i, s_{-i})$ when talking about a player i.

Definition 3. *The distribution is a* correlated equilibrium *if for all players i and all strategies $s_i, s_i' \in S_i$, we have the inequality*

$$\sum_{s_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}} p(s_i, s_{-i}) u_i(s_i', s_{-i}). \quad (6)$$

If player i receives a suggested strategy s_i, the expected profit of the player cannot be increased by switching to a different strategy $s_i' \in S_i$. Nash equilibria are special cases of correlated equilibria, where the distribution over S is the product of independent distributions for each player. Therefore Nash equilibrium is a special case of correlated equilibrium. More precisely, we have the following theorem revealing the relationship between correlated equilibria and Nash equilibria in a 2-player game [16].

Theorem 1. *In any non-degenerate 2-player game, the Nash equilibria reside in vertices of the polytope formed by correlated equilibria.*

Notice that the boundaries of the a polytope are determined by a system of linear equations. While the vertices are characterized as the solutions of various pairs of those linear equations, or equivalently the sparse solution of the differences between those pairs. For example, in $\mathbb{R}^2$, if we have boundaries defined by equations $a_1^T x = 0$ and $a_2^T x = 0$, then a vertex v is a solution of the system $Av = 0$, where $A^T = (a_1, a_2)$. Or equivalently, the solution of the optimization problem $\min\{\|a_1^T v - a_2^T v\|_0 : \|Av\|_2 \leq r\}$, where $0 < r \ll 1$. We then can solve the optimization problem using the compressed sensing theory.

Given all the payoff functions of the 2-player mixing game, we can use the same idea to solve all the vertices of the polytope formed by solution sets of correlated equilibria. By Theorem 1, at least one of the vertices is the Nash equilibrium. We point out the major difference of our method to directly solving system of linear equations is that the latter case requires solving the equations for combinatorically many times since the vertices could be formed by combinatorically many boundaries. While our method only need to perform a joint convex optimization, although at the expense of combinatorically many storage requirement.

We have the following theorem to characterize the usability of our method for solving the Nash equilibria.

Theorem 2. *Assume the payoff matrix $A \in \mathbb{R}^{m \times n}$ satisfies the $(n-1, \sqrt{2}-1) - RIP$ condition, the compressed sensing based method will find the exact Nash equilibria.*

In practice, commonly one is not able to have a precise description of the payoff matrix A. Instead, A is a random matrix *per se*. We can model the uncertainty in the payoff as a Gaussian random matrix, which results A as a Gaussian random matrix. We have the following theorem for the case where A is a Gaussian matrix.

Theorem 3. *Given a Gaussian payoff matrix $A \in \mathbb{R}^{m \times n}$, if with probability at least $1 - \delta$, the matrix $\frac{1}{\sqrt{m}} A$ satisfies the $(i, \epsilon) - RIP$ property provided*

$$m \geq \frac{Ci}{\epsilon} \log\left(\frac{n}{\epsilon^2 i}\right), \quad (7)$$

where $i \geq 1$, $\epsilon \in (0, 1/2)$ and $\delta \in (0, 1)$. C is a constant which only depends on δ. Then with probability p where $p \sim O(e^{1-\delta})$, the compressed sensing based method will solve the exact Nash equilibria.

4. EXPERIMENTAL RESULTS

Compressed sensing methods serve as an efficient way to solve Nash equilibria for certain classed of 2-player games. Therefore, one of the advantages of compressed sensing based method is that it is computationally less expensive than is Lemke-Howson algorithm [17]. To evaluate the performance of our compressed sensing based method, we ran several sets of experiments. In the first set of experiments, we compared the performance of our compressed sensing framework to that of Lemke-Howson algorithm [17] on two 2-play games including battle of the sexes and prisoner's dilemma.

In battle of the sexes game, the husband would most of all like to go to the football game, while their wife would like to go to the opera. Both would prefer to go to the same place rather than different ones. The payoff matrix in Table 1 shows an example of battle of the sexes game, where the wife chooses row and the husband chooses a column. In each cell, the first number represents the payoff to the wife and the second number represents the payoff to the husband.

Table 1. The payoff matrix of battle of the sexes game

	Opera	Football
Opera	3, 1	0, 0
Football	0, 0	1, 3

Both our compressed sensing framework and Lemke-Howson algorithm [17] were executed on this two-player game for 100 times. As can be seen from Fig. 1, the Nash equilibria (blue) are vertices of the correlated equilibria. It it also shown that our compressed sensing framework find the Nash equilibrium $(0.75, 0.75)$ first. Table 2 compares the average computational time and the first Nash equilibrium solution found by these two methods. The table 2 demonstrates that compressed sensing framework solves the game far more quickly than Lemke-Howson.

In prisoner's dilemma game, each player chooses to either "cooperate" or "defect". The payoff matrix in Table 3 shows an example of prisoner's dilemma game, where the player 1 chooses row and the player 2 chooses a column. In each cell, the first number represents the payoff to the player 1 and the

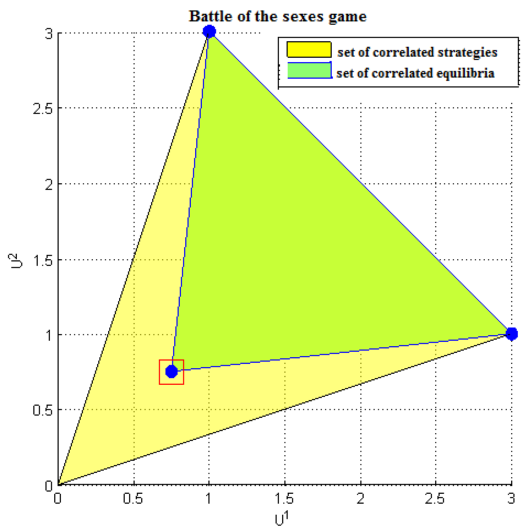

Fig. 1. battle of the sexes game

Table 2. Statistical performance for battle of the sexes game.

Method	NE	CPU time
Compressed sensing	(0.75, 0.75)	0.031
Lemke-Howson	(0.75, 0.75)	1.16

second number represents the payoff to the player 2. The matrix implies that the "both cooperate" outcome is better than the "both defect" outcome.

Table 3. The payoff matrix of battle of the sexes game.

	Cooperate	Defect
Cooperate	4, 4	5, 1
Defect	1, 5	0, 0

Both our compressed sensing based method and Lemke-Howson algorithm [17] were executed on this two-player game for 100 times. Table 4 compares the average computational time and the first Nash equilibrium solution found by these two methods. The table 4 demonstrates that compressed sensing framework solves the game far more quickly than Lemke-Howson.

5. CONCLUSION

In this paper, a compressed sensing based method is proposed to solve Nash equilibria for two-player games. For certain classes of problem, the compressed sensing method provides us with a polynomial-time algorithm. By using the relationship between Nash equilibria and correlated equilibria, we made connections of compressive theory to solving Nash equilibria in dynamic game. It has been demonstrated that our compressed sensing framework can find Nash equilibria with

Table 4. Statistical performance for prisoner's dilemma game.

Method	NE	CPU time
Compressed sensing	(2.5, 2.5)	0.016
Lemke-Howson	(2.5, 2.5)	0.13

known recoverability and stability. We also provide theoretical characterization of the usability of our algorithm. The numerical examples in this paper demonstrate the efficiency of proposed compressed sensing framework in solving Nash equilibria for certain classes of two-player games compared to existing algorithms.

6. REFERENCES

[1] J.A. Filar, K. Vrieze, and O.J. Vrieze, *Competitive Markov Decision Processes*, Springer Verlag, 1997.

[2] A.B. MacKenzie and S.B. Wicker, "Game theory and the design of self-configuring, adaptive wireless networks," *IEEE Communication Magazine*, vol. 39, pp. 126–131, Nov. 2001.

[3] L. Wang and D. Schonfeld, "Game theoretic model for control of gene regulatory networks," in *Proceedings of the 2010 IEEE International Conference on Acoustics Speech and Signal Processing*. IEEE, 2010, pp. 542–545.

[4] L. Brouwer, "Uber abbildung von mannigfaltigkeiten," *Mathematische Annalen*, vol. 71, pp. 97–115, 1910.

[5] C. Lemke and J. J.T. Howson, "Equilibrium points in bimatrix games," *J. Soc. Indust. Appl. Math.*, vol. 12, 1964.

[6] R. Savani and B. von Stengel, "Hard-to-solve bimatrix games," *Econometrica*, vol. 74, pp. 397–429, 2006.

[7] C. Daskalakis, P.W. Goldberg, and C.H. Papadimitriou, "The complexity of computing a nash equilibrium," in *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*. ACM, 2006, pp. 71–78.

[8] L. Khachian, "A polynomial algorithm in linear programming," *SSSR 244: English translation in Soviet Math. Dokl.*, vol. 20, 1979.

[9] F. Facchinei and J.S. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems*, vol. 1, Springer, 2003.

[10] D. L. Donoho, "Compressed sensing," *IEEE Transactions on Information Theory*, vol. 52, pp. 1289–1306, 2006.

[11] E.J. Candès, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *Information Theory, IEEE Transactions on*, vol. 52, no. 2, pp. 489–509, 2006.

[12] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit," *IEEE Transactions on Information Theory*, vol. 53(12), pp. 4655–4666, 2007.

[13] M. Rudelson and R. Vershynin, "On sparse reconstruction from fourier and gaussian measurements," *Communications on Pure and Applied Mathematics*, vol. 61, no. 8, pp. 1025–1045, 2008.

[14] E. Candes, J. Romberg, and T. Tao, "Stable signal recovery from incomplete and inaccurate information," *Communications on Pure and Applied Mathematics*, pp. 1207–1233, 2005.

[15] R.J. Aumann, "Correlated equilibrium as an expression of bayesian rationality," *Econometrica: Journal of the Econometric Society*, pp. 1–18, 1987.

[16] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani, *Algorithmic Game Theory*, Cambridge University Press, 2007.

[17] C. E. Lemke and J. T. Howson, "Equilibrium points of bimatrix games," *Journal of the Society for Industrial and Applied Mathematics*, 1963.